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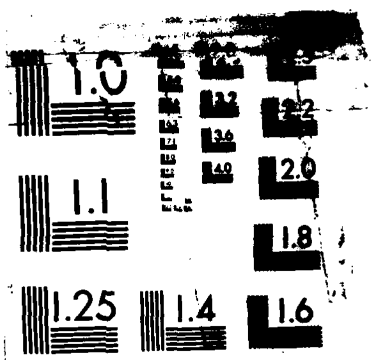

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This report falls naturally into two parts and is written in that form. The first deals with fracture, the second with matters relating to metallic grains.

The object here has been to develop mathematical models, involving singular integral equations, for crack in the presence of plastic deformation and to use these models so as to predict the stress necessary for crack propagation. The quantity of importance in considerations of fracture, namely the crack extension force G (the square of k), follows immediately through the use of the Bilby-Eshelby formula.

Grain size and grain growth are accordingly germane to the study of fracture. A Rayleigh distribution is to be expected on the basis of the random Walker view of grain growth and that is observed experimentally. We conclude then from this agreement and other associated accords that there is strong evidence to support that the mechanism associated with this view is indeed operative and controlling in grain growth.

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FRACTURE MECHANISMS AND METALLIC GRAINS

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This report falls naturally into two parts and is written in that form. The first deals with fracture, the second with matters relating to metallic grains.

PART I FRACTURE

The object here has been to develop mathematical models, involving singular integral equations, for cracks in the presence of plastic deformation and to use these models so as to predict the stress necessary for crack propagation.

It is well known that the processes of plastic deformation in the neighborhood of a crack impedes its advance. It has not been the role of fracture mechanics to provide an understanding of why this should be, but rather to give a workable system of prediction of fracture behavior from certain particular data. To date, considerable success has been achieved by following this route. However, this success has not been uniform and present achievement falls short of the desired goals.

This situation has prompted more fundamental work of which that supported here represents a further stage of advance.

An essential deficiency of fracture mechanics is that the plastic zone at the tip of a crack in a plastically deformable material is treated simply as a region in which elasticity is non-linear. This gives rise to a major conceptual difficulty, the so-called "Rice paradox," namely: that as a consequence of the need to consider an elastic-plastic continuum there is no force available to provide the surface energy of a crack as it advances.

Thus, fracture mechanics gives an imperfect representation of the physics of fracture. Again, we note that it does not take account of the role of dislocations in achieving plastic deformation, nor does it allow for the interactions which occur between these dislocations nor for that between them and the crack.

An immediate consequence of the representation of a plastic zone in terms of dislocations is the realization that there must be a finite region surrounding the crack tip which is free of dislocations. This realization allows the identification of the source of the Rice paradox and it is seen that it is vital towards a thoroughgoing treatment of fracture to include such a dislocation-free zone in the analysis.

As a first exercise in this direction, Chang and Ohr analysed the behavior of a finite planer two-dimensional crack with two plastic zones in the plane of the crack. Thus, the crack was supposed to lie in the plane $y = 0$ in the region $-c < x \leq c$ with plastic zones in the regions $b \leq |x| \leq a$ with $a > b > c$.

Representing equilibrium in this situation in terms of a singular integral equation [1] Chang and Ohr [2,3] were able, in particular, to find the dislocation distribution function and thence the force for crack extension.

The geometry adopted by Chang and Ohr should be appropriate for fracture in modes II or III, but these are not generally situations of technical interest. This deficiency has in part been remedied in the course of the present work.

The essential difficulty encountered in extending this approach to more realistic geometrics, namely those in which crack dislocation are skew one to the other, is that the governing integral equations become much more complex. This difficulty has necessitated the development of a method of solution. This method is, of course, new and its development is a major component of the results of this research. It is detailed below.

Skewed D.F.Z. Cracks in Modes One and Three

The complications in the analysis of D.F.Z. cracks are such that when solutions in closed form are available[2] the results can be expressed only in terms of higher transcendental functions whose significance becomes transparent only after approximations are made. Accordingly, it is germane to consider the use of treatments, which while approximate, are nevertheless quite accurate and allow the analysis to be carried through and the final result expressed in terms of elementary functions. This approach has been used previously for the simplest case[4] where the crack and its associated plastic zones are colinear. It is here used to treat the more complicated and physically more realistic situations in which the plastic zones are skew to the crack plane.

The particular method employed is also applicable to the colinear case where it is most simple. It is as set out below:

We consider a planar two-dimensional crack $-c<x<c$ in a uniform isotropic medium uniformly loaded so as to provide a shear stress σ in the plane of the crack. Plastic zones are supposed to lie in the region $b<x<a$ and to be represented by continuous distributions of dislocations formed in the presence of an applied stress in the four straight lines which make

angles $\pm\gamma$ and $\pm(\pi-\gamma)$ to the plane of the crack. It is supposed that the operative applied stress in these regions is $\sigma-\tau$, $(\tau>\sigma)$.

We calculate and solve,

(a) the dislocation distribution $g(x)$, which vanishes at $|x|=b$, $|x|=a$ and which is developed in the regions $b<|x|<a$ due to a dislocation dipole with elements of strength $\pm f\ (\alpha)$ located at $x = \pm,\ \alpha \leq c$;

(b) the magnitude of a uniform stress $\Delta\tau(\alpha)$ which must be supposed present in order that the dislocation distribution exist;

(c) the stress developed in the region $-c<x<c$ by the distribution $g(x)$;

(d) the resulting singular integral equation for the dislocation distribution representing the crack with the constraint that the implied sum over the elements $f(\alpha)$ gives $\Sigma\Delta\tau(\alpha) = \tau$.

In cases where the distributions representing the plastic zones are skew to the crack length the interactions between dislocation elements are not so simply expressed and this leads to some complications and inaccuracy in the analysis. We commence with an examination of these interactions. Referring to Fig. 1

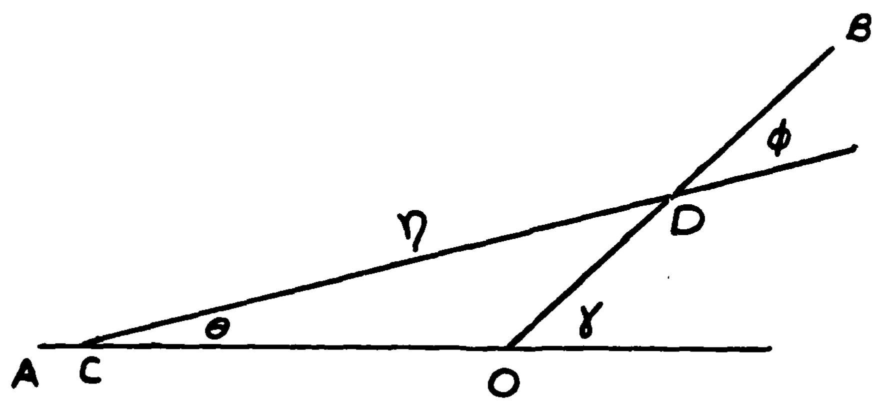

FIG. 1
Crack Tip Geometry

we are interested to find the forces acting in the direction OA and OC on dislocations magnitude λ_1, λ_2 located at the points C and D distant respectively η and ξ from the point O.

We consider two cases, the dislocations are (A) screw and (B) edge.

(A) Screw Dislocation

In this case, the O-B component of the force acting on the dislocation at D is

$$F = A_s \lambda_1 \lambda_2 \cos \phi / \lambda^2 \eta$$

where $A_s = \mu\lambda/2\pi$. From geometry it is readily shown that

$$F = \frac{A_s \lambda_1 \lambda_2}{\lambda^2} \qquad \frac{\xi + \eta \cos \gamma}{\eta^2 + \xi^2 + 2\eta\xi \cos \gamma} \tag{1}$$

$$\simeq \frac{A_s \lambda_1 \lambda_2}{\lambda^2} \qquad \frac{\xi + \eta \cos \gamma}{(\eta + \xi)^2}$$

$$= \frac{A_s \lambda_1 \lambda_2}{\lambda^2} \left\{ \frac{1}{\eta + \xi} - \eta \frac{(1 - \cos \gamma)}{(\eta + \xi)^2} \right\} = \frac{A_s \lambda_1 \lambda_2}{\lambda^2} \left\{ \frac{\cos \gamma}{(\eta + \xi)} + \frac{\xi (1 - \cos \gamma)}{(\eta + \xi)^2} \right\} \tag{2}$$

with a fractional error which is greatest when $\eta = \xi$ and $2\eta\xi(1-\cos \gamma)/(\eta + \xi)^2 = 1$ which then underestimates F by a factor which is 25% when $\gamma = 60°$.

(B) Edge Dislocations

Supposing the Burgers vector to be parallel to the lines on which they are placed, the force which acts along the line of grain between the dislocations is given by

$$F_\eta = \frac{\mu\lambda_1\lambda_2}{2\pi(1-\nu)} \quad \frac{\cos\gamma}{\eta} \equiv \frac{A_e}{\lambda^2} \quad \lambda_1\lambda_2 \frac{\cos\gamma}{\eta}$$

and the contribution to the force for glide along the line OB is

$$F\eta \cos\phi = \frac{A_e\lambda_1\lambda_2}{\lambda^2} \quad \frac{\cos\gamma\cos\phi}{\eta}$$

Additionally, there is a component of force from F_θ

$$F_\theta \sin\phi = \frac{A_e\lambda_1\lambda_2}{\lambda^2\eta} \quad \sin\lambda\sin\phi$$

There results a total glide force

$$\frac{A_e\lambda_1\lambda_2}{\lambda_2\eta} \quad \cos(\gamma-\phi)$$

On the same basis as that used for screw dislocations we find a glide force, e.g. (1)

$$\simeq \frac{A_e\lambda_1\lambda_2}{\lambda^2} \left\{\cos\gamma\left[\frac{1}{\eta+\xi} - \eta\,\frac{(1-\cos\xi)}{(\eta+\xi)^2}\right] + \sin^2\gamma\left[\frac{1}{\eta+\xi} - \frac{\eta}{(\eta+\xi)^2}\right]\right\}$$

$$= \frac{A_e\lambda_1\lambda_2}{\lambda^2} \left\{\frac{(\cos\gamma+\sin^2\gamma)}{\eta+\xi} - \frac{\eta}{(\eta+\xi)^2}(\cos\gamma+\cos^2\gamma-\sin^2\gamma)\right\}$$

or

$$\simeq \frac{A_e\lambda_1\lambda_2}{\lambda^2}\left\{\frac{\cos^2\gamma}{\eta+\xi} + \xi\,\frac{\cos\gamma(1-\cos\gamma)}{(\eta+\xi)^2} + \sin^2\gamma\left[\frac{1}{\eta+\zeta} - \frac{\eta}{(\eta+\xi)^2}\right]\right\}$$

$$= \frac{A_e\lambda_1\lambda_2}{\lambda^2}\left\{\frac{1}{\eta+\xi} + \xi\,\frac{\cos\gamma - \zeta\cos^2\gamma - \eta\sin^2\gamma}{(\eta+\xi)^2}\right\}$$

or better

$$\simeq \frac{A_e\lambda_1\lambda_2}{\lambda^2}\left\{\frac{\cos^2\gamma}{\eta+\xi} + \cos\gamma\,\xi\,\frac{(1-\cos\gamma)}{(\eta+\xi)^2} + \sin^2\gamma\left[\frac{\xi}{(\eta+\xi)^2}\right]\right\}$$

$$= \frac{A_e\lambda_1\lambda_2}{\lambda^2}\left\{\frac{\cos^2\gamma}{\eta+\xi} + \xi\,\frac{(\cos\gamma - \cos^2\gamma)}{(\eta+\xi)^2}\right\}$$

We are now in a position to proceed with the sequence of calculations specified above which leads to a determination of the dislocation distribution for the crack. We consider the case of mode III loading and so screw dislocations.

Distribution Function For Mode III Loading

Referred to ordinary coordinates the force acting on a dislocation, Burgers vector, λ_2 at a point $x = t$ due to a dislocation at $\eta = \alpha$ is from (1):

$$F = A_s\frac{\lambda_1\,\lambda_2}{\lambda^2}\left\{\frac{\rho}{t-\alpha} + \frac{\beta(t-c)}{(t-\alpha)^2}\right\}$$

Hence to facilitate the calculation we make another approximation. We see that

$$\beta\,\frac{(t-c)}{(t-\alpha)^2} \simeq \frac{\beta\,t(t+c)}{2c^2}\,\frac{(t-c)}{(t-\alpha)^2} = \frac{\beta\,t}{2c^2}\,\frac{(t^2-c^2)}{(t-\alpha)^2}$$

with error for a plastic zone size Δ - small compared with the crack length (i.e., $\Delta/c \leq 1$) gives an overestimate of the term in β of at most 5% and partially offsets the underestimate involved in the previous approximation.

On this basis we calculate the dislocation distribution due to a pair of dislocations $\pm\Delta$ located at points $x = \pm\alpha$ together with a uniform applied stress $\delta\tau$. The equation of equilibrium for this distribution is

$$A_s \int_0 g(x) \frac{dx}{x-t} = \delta\tau + A_s \frac{\Delta}{\lambda} \left\{\frac{1}{t-\alpha} - \frac{1}{t+\alpha}\right\}$$

The distribution is chosen to vanish at each of the end points $x = \pm b, \pm a$.

Following Head and Louat [1] we have

$$g(x) = -\frac{\Delta}{\pi^2\lambda} \left[(a^2 - x^2)(x^2-b^2)\right] \int_D \frac{1}{[(a^2-t^2)(t^2-b^2)]^{\frac{1}{2}}} \left\{\frac{\rho}{t-\alpha} + \frac{\beta t}{2c^2}\frac{(t^2-c^2)}{(t-\alpha)^2} + \delta\tau - \frac{\rho}{t+\alpha} - \frac{\beta t}{2c^2}\frac{(t^2-c^2)}{(t+\alpha)^2}\right\} \frac{dt}{t-x}$$

which rearranged, becomes:

$$g(x) = -\frac{\Delta}{\pi^2\lambda} \left[(a^2-\eta^2)(\eta^2-b^2)\right]^{\frac{1}{2}} \int_D \left[\left\{\frac{\alpha\rho+\bar{\beta}t^2(t^2-c^2)}{[(a^2-t^2)(t^2-b^2)]^{\frac{1}{2}}} \frac{d}{d\alpha} \cdot \frac{1}{t^2-\alpha^2}\right\} + \delta\tau\right] \frac{dt}{t-x}$$

where $\bar{\beta} = (1 - \cos\gamma)/c^2$

Here integration is to be carried out over the union of segments $b<|t|<a$ and is easily effected by the use of the appropriate contour and the calculus of residues. We find:

$$g(x) = \frac{\Delta}{\pi\lambda} \left[(a^2-x^2)(x^2-b^2)\right]^{\frac{1}{2}} \left\{\alpha\rho + \bar{\beta}\frac{d}{d\alpha}\right\} \frac{\alpha^2(\alpha^2 - c^2)}{[(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}} \frac{1}{(\alpha^2 - x^2)}$$

provided [4]:

$$\int \frac{1}{[(a^2-t^2)(t^2-b^2)]^{\frac{1}{2}}} \left\{\delta\tau + \frac{A\,\Delta}{\lambda}\left[\alpha + \bar{\beta}\, t^2(t^2 - c^2)\frac{d}{d\alpha}\right]\frac{1}{(t^2 - \alpha^2)}\right\} t^s\, dt = 0;$$

$$s = -\,0,\ 1,\ 2,\ 3.$$

This is satisfied from consideration of symmetry for $s = 0$ and $s = 2$ so that we are left with two conditions associated with cases $s = 1$ and 3. These conditions may be written as:

$$\delta\tau\, \frac{a^2 + b^2}{2} + \alpha\rho k + \alpha\beta k(a^2 + b^2 + 4\alpha^2 - 2c^2)$$

$$= \frac{-\,\alpha^3}{[(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}} \left\{k\rho + k\bar{\beta}\ (5\alpha^5 - 4c^2 - \alpha^2 \frac{(c^2 - \alpha^2)}{(a^2 - \alpha^2)}\right\} \quad , s = 3;$$

$$\alpha^2\delta\tau = \frac{-\alpha^3}{[(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}} \left\{k\rho + k\bar{\beta}\left[3\alpha^2 - 2c^2 - \frac{\alpha^2\,(c^2 - \alpha^2)}{a^2 - \alpha^2}\right]\right\}, \quad s = 1,$$

where

$$k = \frac{2\Delta A_s}{\lambda}$$

We see, for the important range where $\alpha/c = 0\ (1)$, that the right hand sides of these equations are essentially equal. Accordingly, we find that

$$\delta\tau \simeq -2k\alpha \left(\rho + \bar{\beta}\,\frac{(a^2 + b^2 + 4\alpha^2 - 2c^2)}{(a^2 + b^2 - 2\alpha^2)}\right) \qquad (3)$$

provided $\alpha/c \sim 1$, We shall see later that this approximation is adequate for our purposes.

We now determine the stress, in the plane of the crack, developed by the dislocations of the distribution g(x). We have, using an approximation similar to that used before, a force on a dislocation λ, at x, directed along the crack plane, developed from another dislocation λ_2 located at some distance t up the skewed slip plane:

$$F = \frac{A_s \lambda_1 \lambda_2}{\lambda}\left\{\frac{1}{x-t} - \frac{\beta(t-c)}{(x-t)^2}\right\} \simeq \frac{A_s \lambda_1 \lambda_2}{\lambda}\left\{\frac{1}{x-t} - \frac{\beta t}{c^2}\frac{(t^2-c^2)}{(x-t)^2}\right\} \tag{4}$$

Then using (2) and (3) we have at a point x along the crack a stress

$$\sigma(x) = -\frac{\Delta A_s}{\lambda}\left\{\alpha\rho + \beta\frac{d}{d\alpha}\alpha^2(\alpha^2-c^2)\right\}\frac{1}{[(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}}$$

$$\int_D \frac{[(a^2-t^2)(b^2-t^2)]^{\frac{1}{2}}}{\alpha^2-t^2}\left\{\frac{1}{t-x} + \beta\frac{t(t^2-c^2)}{(t-x)^2}\right\} dt$$

where integration is again over the union of the intervals $-a\leq t\leq -b$, $b\leq t\leq a$.

Carrying out this integration we have:

$$\sigma(x,\alpha) = \frac{\pi\Delta A}{\lambda}\left\{\alpha\rho + \beta\frac{d}{d\alpha}\alpha^2(\alpha^2-c^2)\right\}\frac{1}{[(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}}$$

$$\left[1 + \frac{[(a^2-x^2)(b^2-x^2)]^{\frac{1}{2}} - [(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}}{x^2-\alpha^2}\right]$$

$$-\beta\frac{d}{dx}\left\{x^3 + x\left(\frac{[(a^2-x^2)(b^2-x^2)]^{\frac{1}{2}}(c^2-x^2) - [(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}(c^2-\alpha^2)}{x^2-\alpha^2}\right\}\right.$$

Now this is the stress resulting from a single dislocation Δ located at

α. The total stress developed at a point u on the real axis by a dislocation distribution $f(\alpha)$ over the length of the crack is

$$A_s \int \left\{ f(\alpha) \left\{ \frac{1}{u-\alpha} + \frac{\sigma(u,\alpha)}{\Delta A_s} \right\} \right. d\alpha$$

Now since the crack constitutes a free surface this stress together with the applied shear is to vanish.

Then on the basis that the distribution is unbounded at $t = \pm c$ we have [4]:

$$f(u) = \frac{-1}{\pi^2 A[c^2-u^2]^{\frac{1}{2}}} \left\{ \sigma \int_{-c}^{c} [c^2-x^2]^{\frac{1}{2}} \frac{dx}{x-u} - \frac{A}{\Delta} \int_{-c}^{c} [c^2-\eta^2]^{\frac{1}{2}} \frac{d\eta}{\eta-x} \cdot \int_{0}^{c} f(\alpha)\ \sigma(\eta,\alpha)\ d\alpha \right\}$$

Performing the first integral and interchanging the order in the second, this becomes:

$$f(u) = \frac{1}{\pi[c^2-u^2]^{\frac{1}{2}}} \left\{ \frac{\sigma u}{A} - \frac{1}{\Delta} \int_{-c}^{c} f(\alpha) \left\{ \alpha\rho + \beta \frac{d}{d\alpha}\, \alpha^2(\alpha^2 - c^2) \right\} \frac{1}{[(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}} \right. \cdot$$

$$\int_{-c}^{c} \left[1 + \frac{[(a^2-\eta^2)(b^2-\eta^2)]^{\frac{1}{2}} - [(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}}{\eta^2 - \alpha^2} \right.$$

$$- \beta \frac{d}{d\alpha} \left\{ \eta^3 + x \left(\frac{[(a^2-\eta^2)(b^2-\eta^2)]^{\frac{1}{2}}\ (c^2-\eta^2) - [a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}\ (c^2-\alpha^2)}{\eta^2 - \alpha^2} \right. \right\}$$

$$\cdot\ [c^2-\eta^2]^{\frac{1}{2}}\ \frac{d\alpha}{\eta-u} \Bigg]\, d\eta$$

We now introduce the approximation

$$[(a^2-\eta^2)(b^2-\eta^2)]^{\frac{1}{2}} \simeq ab-k\eta^2 \quad , \quad k = ab - [(a^2-b^2)(b^2-c^2)]^{\frac{1}{2}} \tag{5}$$

thereby avoiding the complications associated with elliptic integrals. The error introduced in so doing is estimated (graphically) to be less than 5% and to be such that a reduction of k by such an amount would tend to eliminate that error.

Integrating in (5) we have

$$f(u) = \frac{u}{\pi[c^2-y^2]^{\frac{1}{2}}} \left\{ \frac{\sigma}{A} - \frac{1}{\Delta} \int_{-c}^{c} f(\alpha) \left\{ \alpha\rho + \bar{\beta} \frac{d}{d\alpha} \alpha^2(\alpha^2-c^2) \right\} . \right. \tag{6}$$

$$\left. \frac{k+1 \quad \beta c^2}{[(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}} \left\{ (3-k)\frac{u^2}{c^2} + \frac{3ab}{c^2} - k \right\} d\alpha \right\}$$

Then remembering that we will be concerned only with the behavior of f(u) as $u \to c$ we simplify matters by writing

$$(3-k)\frac{u^2}{c^2} = 3-k$$

in (6).

On this basis we then recognize that (5) can be written as

$$f(u) = \frac{u}{[c^2-u^2]^{\frac{1}{2}}} \left\{ \frac{\sigma}{\pi A} - F(a,b,c) \right\}$$

where F(a,b,c) can be evaluated by substituting this expression for f(u) in (6). We get

$$f(u) = \frac{u}{\pi[c^2-u^2]^{\frac{1}{2}}} \left\{ \frac{\sigma}{A} - \frac{1}{\Delta} \int_{-c}^{c} \frac{\alpha}{[c^2-\alpha^2]^{\frac{1}{2}}} \left\{ \frac{\sigma}{\pi A} - F(a,b,c) \right\}\right.$$

$$\left. \left\{ \alpha\rho + \beta \frac{d}{d\alpha} \alpha^2(\alpha^2-c^2) \right\} \frac{k+1 - \beta c^2}{[(a^2-\alpha^2)(b^2-\alpha^2)]^{\frac{1}{2}}} \left\{ 3-k + \frac{3ab}{c^2} - k \right\} d\alpha \right.$$

$$= \frac{u}{\pi[c^2-u^2)]^{\frac{1}{2}}} \left\{ \frac{\sigma}{A} - F(a,b,c) \right\} \tag{7}$$

Thence we have:

$$F(a,b,c) = \frac{\sigma}{\pi A_s} \frac{G(a,b,c,)}{G(a,b,c)-\Delta}$$

where

G(a,b,c)

$$= \int_{-c}^{c} \frac{\alpha}{[c^2-\alpha^2]^{\frac{1}{2}}} \left\{ \alpha\rho + \beta \frac{d}{d\alpha} \alpha^2(\alpha^2-c^2) \right\} \frac{k+1-\beta c^2}{[(a^2-\alpha^2)(b^2-\alpha^2]^{\frac{1}{2}}} \left\{ 3\left(1 + \frac{ab}{c^2}\right) -2k \right\} d\alpha$$

Once again, to facilitate integration we use the same approximation

$$[(a^2-x^2)(b^2-x^2)]^{\frac{1}{2}} = ab - kx^2$$

and find

$$G(a,b,c) = \frac{\pi D\,[ab]^{\frac{1}{2}}}{k[ab-kc^2]^{\frac{1}{2}}} \tag{8}$$

where D =

$$(k+1 - \beta c^2)(3-k + \frac{3ab}{c^2} - k)$$

Then substituting for G(a,b,c) we have

$$F(a,b,c) = \frac{\frac{\sigma D}{\Delta A_s}\,[ab]^{\frac{1}{2}}\,\{\rho + \beta ab\}}{\left(D[ab]^{\frac{1}{2}}\{\rho + \beta ab\} - k[ab-kc^2]^{\frac{1}{2}}\right)}$$

Here we have neglected terms not involving the factor $(ab-bc^2)^{-\frac{1}{2}}$ which gives a large contribution when the dislocation free zone is relatively small, i.e., when $[c(b-c)^{\frac{1}{2}} >> 1$.

It remains to relate the uniform stress τ which acts to oppose dislocation motion in the intervals $b > |x| < a$ of the plastic zones. Using (3), we replace Δ by $f(x)\delta\alpha$ and integrate over the length of the crack to obtain

$$\tau = -\frac{4A}{\lambda}\int_{-c}^{c} f(\alpha)\alpha\left\{\frac{\rho + \beta(a^2 + b^2 + 4\alpha^2 - 2c^2)}{a^2 + b^2 - 2\alpha^2}\right\} d\alpha$$

$$= -\frac{4AH}{\pi\lambda}\int \frac{\alpha^2}{[c^2-\alpha^2)^{\frac{1}{2}}}\ \frac{d\alpha}{(a^2+b^2-2\alpha^2)}$$

where

$$H = \left\{\rho + \beta(a^2 + b^2 + 4\alpha^2 - 2c^2)\right\}\left\{\frac{\sigma}{A} - F(a,b,c)\right\} \tag{9}$$

Integrating we have

$$\tau = \frac{\mu H}{\pi}\left(1 - \left[\frac{a^2 + b^2}{b^2 + a^2 - 2c^2}\right]^{\frac{1}{2}}\right).$$

This completes the solution. This is not exact since it includes a number of approximations. However, none of these leads individually to serious error. Furthermore, they appear to broadly balance in over-estimating in some cases and underestimating in others so that the net error should be small, i.,e., a few per cent.

The quantity important in considerations of fracture, namely the crack extension force G (the square of k), follows immediately from (7) through the use of the Bilby-Eshelby formula [5].

$$G = \frac{\pi A_s \lambda}{2} \lim_{u \to c} (u-c) f^2(u) .$$

Using this result together with (8) and (9), G can be enumerated for cases of interest.

The Interaction Of Cracks With Cylindrical Microvoids

A characteristic of fracture when significant amounts of plastic deformation occur near the tips of a crack is the appearance of micro-voids. These microvoids are roughly circular in cross section and centered in the plane of the crack. Once these voids are formed further crack extension appears to involve the coalescence of crack and void. Some part of this process is achieved by crack motion. It is, therefore, of interest to examine how the crack extension force (G) is affected by the presence of the microvoid.

Towards this end, we consider a somewhat simplified two-dimensional representation in which the crack is confined to a plane and the microvoid is a cylinder. Specifically, we suppose the crack to lie in the plane $y = 0$, in the range $-c<x<c$ and to be unbounded in the z-direction. We suppose the crack behaves symmetrically so that microvoids are located at each end. These are taken to be cylinders of radius r having their axes parallel to

the z-axis and passing through the points a, o, and -a, o.

In its full generality the determination of G for this geometry is a very complicated process for which it is improbable that an analytical solution is available. However, such generality is not essential here where $r<<c$. Thus, we suppose that the microvoid behaves in the stress field of crack as though it were subjected to a uniform stress σ_a having the value of σ_{yy} developed at the position of the center of the void when the crack alone is present. The approximation so involved has been investigated and shown to be viable [6] for the case where the void is replaced by a thin microcrack. This behavior can be traced to the fundamental feature that the stress variation over the length of a microcrack is for the main part representable by odd functions of distances referred to the center of the microcrack. These odd functions are found to be relatively unimportant in determining the far-field stresses developed by the microcrack. The reason for this characteristic can be traced to the requirement that the total dislocation content of the microcrack be zero. A result of this constraint is that the far-field stresses exhibit quadrapolar for odd, as opposed to dipolar for even, functional behavior. Since the same constraint must apply for microvoids we can expect them to exhibit similar behavior.

Again, it is possible to make a simple examination of the effects of the stress components representable by even functions. Thus, we write the stress in the neighborhood of a crack under a uniform stress σ as:

$$\Sigma = \frac{\sigma\eta}{[\eta^2-c^2]^{\frac{1}{2}}} \equiv \frac{\sigma(a+t)}{[(a+t)^2-c^2]^{\frac{1}{2}}} . \qquad (11)$$

Thus, we suppose the void to be centered at $\eta = a$ and that the variable t measures distances from that point. Expanding this expression in a Taylor series we find a mean stress over the interval $-\Delta<t<\Delta$:

$$\bar{\Sigma} = \frac{\sigma a}{[a^2-c^2]^{\frac{1}{2}}} \left\{ 1 + \frac{c^2\Delta^2}{2(a^2-c^2)^2} + \frac{3c^2}{4!} \frac{(4a^2+3c^2)}{(a^2-c^2)^2} \Delta^4 + \cdots \right\}$$

For $\Delta <\sim \cdot 7(a-c)$ and $c \sim a$ this series is rapidly convergent and $\bar{\Sigma}$ nearly constant and equal to

$$\Sigma = \frac{\sigma a}{[a^2-c^2)]^{\frac{1}{2}}}$$

which is the stress calculated from (1) at the position of the center of the void. For the reasonable case where $\Delta = r = a-c/2$ the actual mean value deviates from Σ by only about 3%. Accordingly, we proceed as though the microvoid were subjected to the uniform stress σ_a.

Now, if the dislocation distribution which represents the rate of change of surface displacements over the crack surface is f(x), the stress normal to the plane of the crack at a point t,o due to the distribution, is given by

$$-A \int_{-c}^{c} f(x) \frac{dx}{x-t} = \sigma(t). \tag{10}$$

Here $A = \mu\lambda/2\pi(1-\nu)$, μ is the shear modulus for the material assumed isotropic, ν is Poisson's ratio and λ represents the magnitude of a unit dislocation. The total stress at t,o is then

$$g(t) = \sigma(t) + \sigma$$

where σ is the applied stress and the mean value of g(t) over the length 2r of the void is

$$\frac{1}{2r}\int_{-c}^{c} g(t)dt = \bar{g}(a) \simeq g(a)$$

Again, in the presence of a uniform normal tension $\tau_{yy} = \tau$ the stresses developed by a pair of cylindrical holes radius r centered at $x = \pm a$ is at points x,o:

$$\tau(x) = \frac{\tau}{2}\left\{\frac{r^2}{(x-a)^2} + \frac{3r^4}{(x-a)^4} + \frac{r^2}{(x+a)^2} + \frac{3r^4}{(x+a)^4}\right\} \tag{12}$$

We then suppose that the factor, $\tau \equiv \bar{g}(a)$, is known and in the first instance determine the distribution function which arises from the action of the stress

$$\sigma + \tau(x)$$

over the length of the crack.

Following Head and Louat [1] we have the distribution

$$f(x) = -\frac{1}{\pi^2 A}\frac{1}{[c^2-x^2]^{\frac{1}{2}}}\int_{-c}^{c}[(c^2-t^2)^{\frac{1}{2}}(\sigma+\tau(t)]\frac{dt}{t-x}.$$

In the evaluation of this integral, it is helpful to note that we can write

$$\tau(x) = \frac{\tau}{2}\left(x^2\frac{d}{da} + \frac{x^4}{4}\frac{d^3}{da^3}\right)\frac{2a}{x^2-a^2}$$

Substituting for $\tau(x)$ from (12) in (13) and integrating by considering the contour integral

$$\int (z^2-t^2)^{\frac{1}{2}}[\sigma+\tau(z)]\frac{dz}{z-x}$$

we find

$$f(x) = \frac{x}{\pi A[c^2-x^2]^{\frac{1}{2}}} \left\{ \sigma - \bar{g}(a)x^2 \left(1 + \frac{x^2}{4}\frac{d^2}{da^2}\right) \left(2a \frac{(2c^2-a^2-x^2}{(a^2-c^2)^{\frac{1}{2}}(a^2-x^2)^2}\right) \right\}$$

Then according to (1) the stress at x is given by

$$= \frac{1}{\pi} \int_{-c}^{c} \frac{x}{[c^2-x^2]^{\frac{1}{2}}} \left\{ \sigma - \bar{g}(a)(x^2) \left(1 + \frac{x^2 d^2}{4da^2} \right) \left(- 2a \frac{(2c^2-a^2-x^2)}{(a^2-c^2)^{\frac{1}{2}}(a^2-x^2)^2} \right) \frac{dx}{x-a} \right. .$$

Here the evaluation of the contour integral is complicated by the presence of a triple poles at $z = a$ and a double pole at $z = -a$. Nevertheless, the algebra is straightforward if somewhat lengthy and we find a total stress

$$\sigma = \frac{a}{[a^2-c^2]^{\frac{1}{2}}} + \bar{g}(a) \frac{r^2c^2}{4} \left(1 + \frac{r}{4}\frac{d^2}{da^2} \right) \frac{2a^2-c^2}{a^2(a^2-c^2)^2}$$

But by definition this is equal to $\bar{g}(a)$. As we saw above, if $a-c/c$ is small, we may put $g(a)$. On this basis we find

$$g(a) \frac{4a^2(a^2-c^2)^2 - r^2c^2 (2a^2-c^2)}{4a^2(a^2-c^2)^2} = \sigma \frac{a}{[a^2-c^2]^{\frac{1}{2}}} ,$$

so that

$$g(a) = \frac{\sigma a}{[a^2-c^2]^{\frac{1}{2}}} \frac{4a^2(a^2-c^2)^2}{\left[(4a^2(a^2-c^2)^2 - r^2c^2(2a^2-c^2) \right]} .$$

We now write $a = c + nr$ where $nr/c \ll 1$ and obtain:

$$g(a) \simeq \frac{\sigma a}{[a^2-c^2]^{\frac{1}{2}}} \cdot \frac{1}{\left[1 - \frac{1}{16n^2}\right]^2}$$

and have from (4) and with $\bar{\tau} = \bar{g}(a)r^2$

$$f(x) \simeq \frac{x\sigma}{\pi A[c^2-x^2]^{\frac{1}{2}}} \left\{ 1 - \frac{r^2 a}{(a^2-c^2)} \frac{2c^2-a^2-x^2}{(a^2-x^2)^2} \cdot \frac{a}{1-s} \right\},$$

where $s = \dfrac{1}{1 - \frac{1}{16n^2}}$

The evaluation of G then follows from Bilby and Eshelby's result [5]

$$G = \frac{\pi A \lambda}{2} \lim_{x \to c} (x-c)\, f(x)^2$$

Substituting we obtain

$$G \simeq \frac{\pi\sigma^2 c(1-\nu)}{2\mu} \left[1 + \frac{1}{4n^2} \frac{1}{1-s}\right]^2$$

$$\simeq \frac{\pi\sigma^2 c(1-\nu)}{2\mu} \left(1 + \frac{1}{2n^2}\right)$$

so that G is increased by ~ 10% when n = 2 and the cylinder is centered at a diameter distance from the tip of the crack.

Surface Displacement and Dynamic Fracture

Surface displacement near the tip of a crack has become of importance with the use of the so-called caustic method of examining crack behavior.

In this method the lateral surface of a fracture surface is polished to a mirror finish. The area adjoining a crack tip which is normal to this surface is observed using a system of optics which responds to changes in inclination of the surface. Such inclinations are induced in the surface as a consequence of elastic and plastic deformation and result in the appearance of a dark area bordering the crack tip. This dark area or rather its boundary is the caustic. It has been found experimentally that this curve approximates to a circle. This description is found to be increasingly apt as the optics are adjusted so that the caustic curve is generated at distances from the crack tip where deformation is essentially elastic.

Again, it has been found that in these circumstances that there is a one to one correspondence between the crack extension force applied (as determined from K_{IC}) and the radius of the caustic. This allows of a simple method of measuring this quantity.

This result has been derived theoretically for the case of static cracks and also for a dynamic crack, providing its speed of advance is significantly less than that of sound.

The method of analysis rests on the representation of a crack in mode I as an array of edge dislocations in the plane of the crack. The distribution of this array is heavily concentrated toward the tip. Thus, to a first approximation the crack may be represented as a single giant dislocation located at the crack tip.

Now, adopting a coordinate system in which the axis of the dislocation is the z-direction while the plane of the crack is represented by y = o, we have for an infinite system a stress

$$\sigma_{zz} = \frac{\mu b(1+\nu)}{2\pi(1-\nu)} \frac{y}{x^2+y^2} .$$

Here μ is the shear modulus, ν is Poisson's ratio and b is the Burgers vector of the dislocation. When such a dislocation terminates at a free surface it is displaced so as to generate stresses which just cancel those due to the dislocation. These displacements are accompanied by inclinations which are constant along lines where the stress is constant. It is a simple matter to show that this is the case along a circle whose center lies in the plane of the crack and whose circumference passes through its tip.

Thus, theory accords with experiment in confirming the usefulness of the caustic approach.

PART II GRAIN BOUNDARIES AND OSTWALD RIPENING

It has long been recognized that, other things being equal, materials in which grains are finer are tougher than those where the grains are coarser. Grain size and grain growth are accordingly germane to the study of fracture. Furthermore, the rate of growth of grains of a particular average size is a strong function of the size of particles of another phase which lie in the grain boundaries. Broadly, for a constant volume fraction the restraint offered by such particles decreases as the size of the particles increase through Ostwald ripening. Again, there is evidence available that the processes of normal grain growth and of Ostwald ripening (at non-vanishing concentration of second phase) are both stochastic in origin.

Accordingly these may be regarded as dual topics. Much of what can be said for one is applicable to the other. We commence with grain growth.

Grain Boundaries

We remark that in a polycrystal not all grains are of the same size and, in fact, grain size is found to be distributed in a way which, remarkably, is independent of mean grain size. We now note that such a state of affairs could be expected if during grain growth (or Ostwald ripening) grains make excursions in size which are proportional to the mean and which have equal probability of resulting in an increase as a decrease in size.

On this basis if the fraction of grains of size lying in the range l to $l+\delta l$ be $f(l)\ \delta l$, we find a governing equation

$$\frac{\partial f}{\partial t} = A \frac{\partial^2 f}{\partial l^2} \tag{14a}$$

where A is constant.

For this an appropriate solution is

$$\partial(l) = \frac{1}{A l^{3/2}} \; l \; e^{-l^2/4At}$$

for the case of Ostwald ripening similar considerations hold. These will be detailed later.

We first consider how this approach accords with the whole body of evidence relating to grain growth.

In the process of normal grain growth some grains of a polycrystalline aggregate vanish, without creating voids. The volume thus made available is distributed among the remaining grains, resulting in an increase in their mean size; in other words, in grain growth.

Salient features of this phenomenon, and ones with which any acceptable theory must be in accord, are that the processes are essentially steady state in that the distribution of grain sizes is invariant with the amount of growth which has occurred, that the rate of grain growth varies as t^n, where $-1 < r \leq -.5$, that individual grains are oriented at random, and that the grain structure is equi-axed.

Measurements relating to normal grain growth have, for the main part, been restricted to those which can be made using planar sections. In particular, these allow the determination of the distribution of grain sizes and of the number of faces to a grain, the rate of grain growth and its dependence on time. Similar results have also been obtained from computer modeling. It is required of theory that it give correct qualitive predictions of the results and also those obtained when the measurements relate to three dimensions.

A first task in delineating the requirements on theory is to decide just what is the mathematical form of the distribution of grain size, l. In the past, this distribution has been fitted to four different functions: lognormal[7], the Rayleigh function [8,9], a complex exponential form[10] and that of Lifshitz and Slyozov[11,12]. These functions are respectively:

$$g(l) = A\, e^{-\ln^2(l/l_0)/2\sigma^2} \quad ; \tag{15}$$

$$f(l) = B\, l\, e^{-(l/l_1)^2} \quad ; \tag{16}$$

and

$$h(l) = c\, e^{-D([l]^{\frac{1}{2}} - d)^2} \quad ;$$

$$k(l) = \frac{\beta u}{(2-u)^2 + \beta} \cdot (2e)^{\beta}\, e^{-2\beta/2-u} \quad , \quad u = l/l_0 \tag{17}$$

β is equal to the dimensionality involved, i.e. 2 or 3.

Here A, B, C, D, l_0, and l_1 are disposable constants.

No one of these equations gives a good fit to all the data. The best individual fit is probably that of (17) applied to the data given by Aboav and Langdon [13]. Again, it would seem that this equation is capable, through adjustments to the parameter D, of giving a reasonably acceptable fit to all the data. However, since any theoretical justification seems improbable, it is difficult to accept that this agreement is other than fortuitous. The difficulty with this form is essential and lies in the fact that the distribution does not tend to zero at zero grain size [8].

In recent years there has been a growing conviction among many workers that the distribution of grain sizes approximates to the lognormal form. This conclusion usually rests on the following observations. Data is plotted using special graph paper in which the abscissa is the cumulative percentage of grains measured. This scale is non-linear and given by

$$F(x) = \frac{\int_x^{\infty} e^{-(\ln d/l_0)^2/2\sigma^2} \, dl/l}{\int_0^{\infty} e^{-(\ln l/l_0)^2/2\sigma^2} \, \frac{dl}{l}}$$

The ordinate is scaled as lnx. Data which obeys a lognormal distribution then plots as a straight line with slope, σ.

Thus, an examination of such plots shows a small but definite deviation from the straight line configuration. Data are, as is frequently recognized, only approximately lognormal. Consistently, data plotted directly against the logarithm of the grain size shows significant assymetry about the peak. Again, a direct comparison of the relations (2) and (3) shows that they can be in essential accord if the constants B, C, D and have suitable values. It was shown that the two are in reasonable agreement when $\sigma \sim .5$. This suggests that a plot of data which actually follow the Rayleigh distribution would approximate to a straight line when plotted on the special graph paper mentioned above. Clearly, the data of Conrad, Swintowski and Mannan [14] do conform approximately to straight lines which do not deviate far from the curves. Nevertheless, the latter provide a better fit to the data than do the lines. The standard deviation in the lognormal representation of these results approximates to .6 and it is apparent that this is the case for most published material. Thus, Schukler[15] lists results from eight materials. One of these α-iron, is described as not not being lognormal, the remaining seven have standard deviations which average .59 and which lie in the range 0.468 to 0.674. We conclude that all experimental results which give standard deviations of about this amount on the lognormal approximation will be in reasonable agreement with the Rayleigh distribution. This is not to say that all grain size distributions describable as lognormal can also be considered to be Rayleigh. For example, Rhines and Patterson [16] have found distributions in which $\sigma \sim 2$. Data approximating to a lognormal distribution for standard deviations in this range is not simply describable in terms of Rayleigh. However, it should be borne in mind that these data cannot be expected to be representative of normal grain growth since they were obtained using pre-strains in the range 2 - 30%; values so low as to give non-uniform nucleation during recrystallization.

Passing now to a consideration of grain shape, that is the distribution of a number of faces, or in two dimensions, of edges per grain, it is clear that here the form is lognormal or a very good approximation to it. All authors, including Hu, who found that his grain size distribution data was not lognormal, appear to agree on this point. This discrepancy is not at variance with the several relations which have been found between the diameter of a grain and the number of its sides:

$$n = 3 + kl$$

where n is the average number of sides of per grain of diameter l[2]; and

$$N = 2 + KL$$

where L is the average diameter of a grain with an integral number of sides[6]. If all grains having N sides were the same size, it would be expected that grain size and shape would have the same distribution. As it is, they should be similar but not identical.

Allowing that there is this difference, no explanation has yet been advanced as to why the distribution of shape should be lognormal. Toward such an explanation we recognize that observations of grain shape derive from planar intersections of three-dimensional ensembles. In contradistinction with grain size there is no unique transform relating characteristics in two to those in three dimensions. Thus, the shape observed as a consequence of a planar intersection with one particular grain depends not only on the number of its faces but also on their sizes and orientation with respect to the intersecting plane; that is to say, a multiplicity of transforms are operative. Then allowing, as has been shown[3], that if the distribution of grain size is Rayleigh in two dimensions it is Rayleigh in three also, we see that the distribution of shape in two-dimensions cannot be Rayleigh if, as seems reasonable, we suppose it is in three. We now suppose that the observed shape distribution is a linear superposition of different and independent distributions each derived from the operation of a different transform operating on the distribution of shape in three dimensions. Each of these distributions must accord with the Euler rules so that the average number of sides to a grain must be 6. Again, supposing that no grain have less than three sides, we require that each distribution vanish when n, the number of sides, is 2 and when it is infinity.

Now, according to the central limit theorem of probability theory these characteristics are consistent with the appearance of a lognormal distribution and it is to this that we address its appearance.

The Rayleigh distribution was derived on the basis that grain growth is a stochastic process. Besides the accord between theory and experiment indicated above, additional support for this approach has recently become available as the result of computer modeling of grain growth. This support comes in four ways:

The calculated distribution is in good agreement with the Rayleigh nearly everywhere; the distribution is found to be the same in three as it is in two dimensions. This is critical* since it is a unique property of the Rayleigh function that it is invariant to the Abelian transform used to generate the distribution function in three dimensions from that in two, or vice verse;

Consideration of the trajectories of individual grains in grain size space shows that size and growth rate are not directly related. Furthermore, as postulated, grains do show changes in the direction of their growth even though they are much larger than the mean grain size.

Finally, this work also indicates a linear relation between the number N of sides of a grain and the mean diameter L of grains of this class. Thus,

$$N = 2 + KL$$

The constant, K was evaluated and found to be 0.23. Assuming that such a

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* Other functions will, however, satisfy this transform to various degrees of approximation.

relation exists, the value of K can be evaluated from the Rayleigh distribution on the basis that the average corresponds to grains with six sides[17]. The value predicted in this way is 0.25.

Thus, there is now a significant amount of evidence to support the view that random walk is of dominating importance in grain growth. Accordingly, we shall examine its basic features so as to determine whether such behavior should be anticipated. We shall then be concerned to justify the couchment of resulting analysis in terms of the linear dimensions of the grains rather than, say, their volume.

In this connection, the first matter to be settled is whether grain growth is to be regarded as involving the grains in and of themselves or simply the interfaces between them. We are inclined to the view that since only the boundaries move during grain growth the presence or absence of the material within the grain, which makes up the grains, is largely, if not entirely, irrelevant. Thus, we expect the constraint of constant volume (area in two dimensions) while important and necessarily included, need not be central to the theory. Rather, we should include those constraints which arise from the fact that we are concerned with an assembly of faces in quasi-equilibrium under the forces due to their intrinsic energy; that is to say, we take due note of the so-called Euler rules[17].

In three dimensions, these require that 3 faces meet in a line and four such lines meet at a point forming a four-fold node. In two dimensions, three edges must meet at a point. On this basis and with the additional constraint that edges at a node are spaced 120° apart, Von Neuman[18] examined the behavior of an n-cornered cell of a two-dimensional bubble froth. From his determination of the resultant forces, it can be inferred[12] that the cell will tend to grow at a rate

$$\frac{dl}{dt} = M\,\frac{n-6}{n} \tag{18}$$

where M is a constant, n the number of corners (or sides) and l a measure of the linear dimensions of the cell. A comparable expression applicable to three dimensions has yet to be determined, but it can be expected that a somewhat similar equation should hold there, perhaps

$$\frac{dl}{dt} = K\,\frac{m-24}{m} \tag{19}$$

where m is the number of grain edges. The numbers 6 and 24 of eqs. (18) and (19) are analogous. In two dimensions, an array of grains obeying modified Euler rules is stable when each grain has six corners[17]. In three dimensions a stable space-filling ensemble of polyhedra is found[19] when each has 14 faces, 24 edges and 36 apices. Thus, much if not all the physics of grain boundary structure essential to our purposes is encapsulated in (18) and (19) and it is important to stress that these equations relate numbers specifying a topological quantity to a linear dimension. Allowing that grain growth involves an increase in size of the average grain of an assembly, it is apparent that such growth can occur only as a result of the disappearance of particular grains. We can as direct consequence of the modified Euler conditions identify these disappearing grains as tetrahedra (triangles in two dimensions). Thus, a tetrahedron is unique among polyhedra in having four apices. Migration of these apices to a central point leaves, as required for stability, a four-fold node. This fact, together with (19) would seem to suggest a model in which a grain with fewer than 24 edges (six corners) would progressively lose them until it finally disappears, while conversely, grains with more than this number grow indefinitely. These features together with the seemingly reasonable assumption that grain diameter is proportional to n forms the basis for the treatment given by Hillert[12] and expressed by either of the equations:

$$\frac{dl}{dt} = a\left(\frac{1}{l} - \frac{1}{\bar{l}}\right) ; \qquad (20)$$

$$\frac{dn}{dt} = b\left(\frac{1}{n} - \frac{1}{6}\right) ;$$

where a and b are constants. $\bar{l}$ is the diameter of a six sided grain.

An immediate difficulty with this approach is that it takes no account of grain contiguity; an omission which we now attempt to rectify. In the interests of simplicity we restrict the discussion to considerations in two dimensions. We observe that while von Neumann's analysis indicates whether a particular grain will tend to grow or to shrink, it does not bear on the question as to whether or not a particular side will extend or contract, possibly vanishing. This possibility arises because each side is (1) shared by two grains and (2) terminates in two grains. Toward a determination of the consequences of these features, we first consider the well known change in configuration illustrated in Fig. 2.

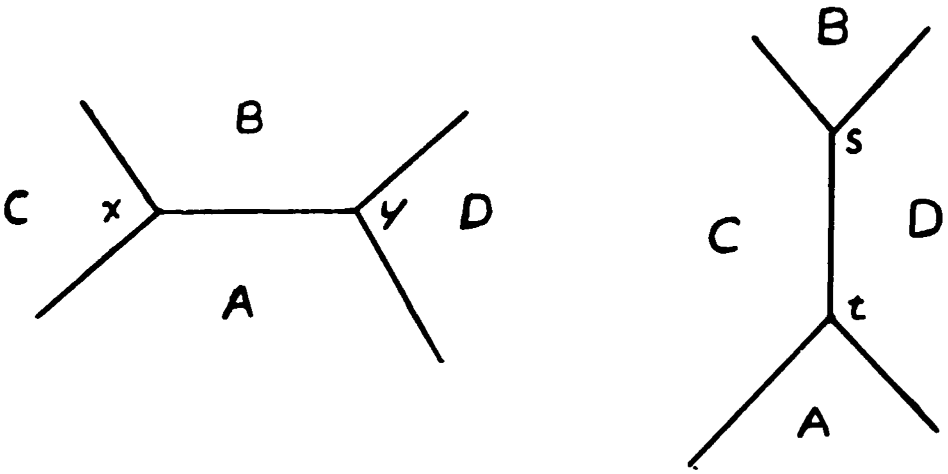

FIG. 2
Illustrating a change in the grain configuration in two dimensions.

Here the merger of the grain corners x and y and their replacement by ρ and t is accompanied by the loss of one of one side, by each of the grains A and B and an equal gain by both C and D. The converse is true if ρ and t merge. As demanded by Euler's rule there is not change in the number of sides in either case. Thus, a grain, for example A, may either gain or lose a side by one or other of the procedures. A question, crucial in the assumption expressed in (6) now arises: to what extent is the choice between these alternatives determined by the characteristics of grain A? Toward a resolution of this question, we first examine the average behavior of an n-sided grain and refer to the grain corner illustrated in Fig. 2. Here the lines represent the chords joining the ends of grain edges which are taken to be bent into arcs of circles [18].

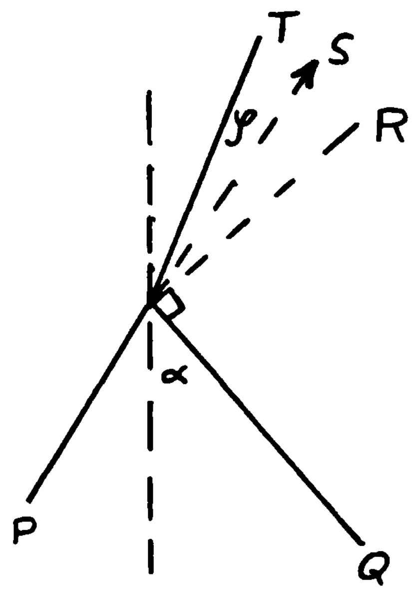

FIG. 3
Grain Corner Geometry

We consider the cases where n > 6 so that α, the mean value α, is greater than $\pi/6$. We see that as shown, the point O is not at a position of equilibrium and consequently that it will move in response to the line tensions T directed along the directions OP, etc. as shown. If the resultant motion in the direction OS lies within the right angle QOR, the grain edge will tend to shorten; if it lies within the angle ROT, the edge will lengthen. Resolving forces, the condition for shortening is easily shown to be

$$\frac{T \sin \gamma}{T \cos \gamma - 2T \cos \alpha} > \cos t\alpha.$$

When rearranged this becomes

$$\cos(\gamma + \alpha) < 2 \cos^2\alpha. \qquad (21)$$

This condition is satisfied except when γ may be regarded as small. Thus, when n = 7, $\alpha = 64.29$ we find that on average the side OQ tends to shorten unless $y<y_c$, $y_c = 3.59$.

The most restrictive case occurs when $y_c = y_{max}$, the grain in POQ is 12-sided and $y_c = 7.5^0$. Similar considerations can be applied to the case where grains have less than six sides. There, we find, when α takes its mean value, $\bar{\alpha}$, the sides OQ move so as to shorten, for all y.

It is helpful in appreciating the implications of these results to consider the case of a six-sided grain in the special circumstances where each angle POQ = 2α has the mean value $2\pi/3$. Here (21) is satisfied for all positive γ and fails when γ is negative. But at each grain corner, there are two grain sides to be considered. In these two cases γ takes equal and opposite values. Accordingly, one side shrinks, the other lengthens. Then since each edge has two ends we see that four possible situations can arise. In one, both ends tend to move together, in another they move apart, in the remainder the motions oppose. If then, as seems to be indicated we assign equal probability to each case we see that the chance of a side lengthening is just equal to that of its shortening. Thus, as might be expected, we find that a six-sided grain has equal probility of tending to gain as to lose sides.

In the case of grain with more than six sides there is bias toward the extension of all sides. The amount of this bias is

$$\gamma_c/\bar{\theta} = R$$

$\bar{\theta}$ is the range of θ (see Fig. 2) found from an appropriate weighting for the frequency of occurence of particular values. Here it is sufficient to recognize that θ can lie in the range $60^0 \sim 150^0$ and hence, in view of (21), that R should be significantly less than 1 in all cases in which $n>6$. We conclude, then, that the processes by which grains gain and lose sides during growth are in the main random rather than directed when $n>6$.

When, on the other hand, $n<6$ the situation is reversed. With decreasing n there is then a steadily increasing correlation between a change in grain size and the number of its sides. This is so because (21) is satisfied for all γ when α takes its mean value $\bar{\alpha}$, since $\bar{\alpha}<\pi/3$. Thus, (21) can fail and the grains have a tendency to gain a side only at those corners at which stochastic variation provides a value of α exceeding $\pi/3$. Such excursions can be expected at reasonably common when $n = 5$ and $\bar{\alpha} = \pi/2$ but rare when $n = 4$ ($\bar{\alpha} = \pi/6$). However, Imam,[20] using a photoemission electron microscope to observe grain boundary motion in titanium has reported a sequence in which a shrinking four-sided grain gained a side before losing it again.

Thus, we find that when $n>6$, the path of addition (and subtraction) of grain sides should be essentially that of a random walker. The dominance of directed "flow" by stochastic factors is continuously reduced and finally disappears as n decreases below 6.

The effect of this variation has been examined by Pande[19] who has modeled this system by considering the equation

$$\frac{\partial f}{\partial t} = A \frac{\partial^2 f}{\partial l^2} + B \frac{\partial (f/l)}{\partial l} .$$

Here the second term represents the effect of directed flow. This equation has a closed form solution. For reasonable values of the ratio B/A the distribution deviates only slightly from that calculated when B = 0, namely that of Rayleigh.

We see then that the Rayleigh distribution is to be expected on the basis of the random-walker-view of grain growth and that is observed experimentally. We conclude, then, from this agreement and other associated accords that there is strong evidence to support that the mechanism associated with this view is indeed operative and controlling in grain growth.

Ostwald Ripening

In place of (14a) we have, again a basis of random excursions:

$$\frac{\partial f}{\partial t} = B\,\overline{\Delta x}^{\,2}\,\frac{\partial^2 f}{\partial x^2}$$

where $x = 4\pi r^2$, r is the radius of a particle, assumed spherical and B is a rate factor. f(x) is now the distribution of surface areas of the particles. Surface area rather than particle radius is adopted (without prejudice to the result) as being the more physically realistic quantity.

To fix B we recognize that surface displacements scale with the size of the surface and thus with r^2 while the rate of change of particle radius varies as $1/r$.

Thus,

$$B \sim \frac{1}{\bar{r}^3} \propto \frac{1}{\bar{x}^{3/2}}\,, \tag{22}$$

where $\bar{r}$ and $\bar{x}$ are the mean values of r and x, and (22) becomes:

$$\frac{\partial f}{\partial t} = A\,\bar{x}^{\frac{1}{2}}\,\frac{\partial^2 f}{\partial x^2} \tag{23}$$

where A is a constant.

Then supposing $\bar{x} \simeq \bar{x}(t)$ and that

$$\tau \equiv \int_0^t x^{\frac{1}{2}}(t)\,dt \tag{24}$$

(23) becomes

$$\frac{\partial f}{\partial \tau} = A\,\frac{\partial^2 f}{\partial \bar{x}^2} \quad ,$$

with a solution

$$f = \frac{x}{\tau^{3/2}}\;e^{-\frac{x^2}{4A\tau}} \tag{25}$$

Then, rate of growth of $\bar{x}$ is as $\tau^{\frac{1}{2}}$ and

$$\frac{d\tau}{dt} = \bar{x}^{\frac{1}{2}}\;\alpha\tau^{1/4},\; \tau^{3/4}\alpha\; t,\; \tau^{\frac{1}{2}}\alpha\; t^{1/3}$$

Thus we conclude on this basis that Ostwald ripening proceeds as the one-third power of the true. This accords with experiment.

Again, expressed in ordinary time units and particle radius, the distribution becomes

$$f = \frac{r^2}{t}\;e^{-\left(\frac{r}{t^{1/3}}\right)^4/4A}$$

This distribution has been found to accord well with available experimental evidence.

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